

Classification of Minimal Separating Sets in Low Genus Surfaces

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Abstract

Consider a surface S and let $M \subset S$. If $S \setminus M$ is not connected, then we say M *separates* S , and we refer to M as a *separating set* of S . If M separates S , and no proper subset of M separates S , then we say M is a *minimal separating set* of S . In this paper we use methods of combinatorial topology to classify the minimal separating sets of the orientable surfaces of genus $g = 2$ and $g = 3$. The classification for genus 0 and 1 was done in earlier work, using methods of algebraic topology.

Surfaces and Separating Sets

We begin with some basic definitions.

Definition 1. *When we say surface we are referring to a closed, connected, triangulated, orientable, 2-manifold.*

Let S be a surface and let $M \subset S$.

Definition 2. *If $S \setminus M$ is not connected, then we say M separates S , and we refer to M as a separating set of S .*

Definition 3. *If M separates S , and no proper subset of M separates S , then we say M is a minimal separating set of S .*

In this paper we will not consider pathological separating sets, such as the common boundary of the Lakes of Wada. To that end, when we refer to subsets of a surface it will be understood that we are referring to finitely triangulated subsets. Such subsets may be thought of as finite simplicial subcomplexes of the triangulated surface. Such sets occur naturally in the following geometric context. Let S be a connected compact surface whose distance $d(.,.)$ is inherited from a Riemannian metric. Given two distinct points p and q in S , the mediatrix is the set of points x such that $d(x, q) = d(x, p)$. In [2] and [5] it is proved that mediatrices are closed simplicial 1-complexes. Furthermore the topological classification of minimally (triangulable) sets up to graph isomorphism was given for the (oriented) surfaces with genus 0 and 1. Namely, the 2-sphere admits

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only one minimal separating set (a simple, closed loop), and the torus admits 5 topologically distinct minimal separating sets.

Here we carry out the classification up to graph isomorphism for the (oriented) genus 2 and 3 surfaces. Our proofs differ substantially from those in [2] and [5] where methods of algebraic topology were used. In this paper we use combinatorial topology and combine it with a substantial computational effort to complete the classification.

Graphs and Embeddings

Colloquially speaking, when we talk about *graphs* we are referring to undirected multigraphs which may or may not have loops. This is made rigorous as follows.

Definition 4. A graph G is a pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set of elements called vertices, and E is a finite multiset whose elements are unordered multiset pairs of vertices in $V(G)$. Elements of $E(G)$ are referred to as edges and the elements of edges are referred to as edge-ends. Two graphs G and H are the same ($G = H$) if there is a bijection from $V(G)$ to $V(H)$ that preserves the edges.

When the graph in question is clear from context, we may use the simpler notation V and E to refer to the vertex set and edge set, respectively, of a given graph.

Definition 5. A vertex contained in an edge is said to be incident to the edge.

Definition 6. Vertices $v_1, v_2 \in V$ are said to be adjacent if there exists an edge that contains them both.

Definition 7. An edge whose edge-ends are identical is called a loop. An edge that is not a loop is called a non-loop.

Definition 8. Let v be a vertex of a graph. Let l and n be the number of loops and non-loops, respectively, incident to v . Then the degree of v , denoted $\deg(v)$, is given by $n + 2l$.

Informally, the degree is the number of edge-ends incident to the vertex.

Definition 9. Let G be a graph and let $A \subset V$. Then the subgraph of G induced by A , denoted $G[A]$, is the graph whose vertex set is A , and whose edge set is the subset of $E(G)$ all of whose edge-ends are in A .

Less formally, the induced subgraph $G[A]$ is the graph obtained from G by removing all vertices from G except those in A . Edges that lose an edge-end during this process are also removed.

We often want to consider what parts of G remain after the removal of some subset B of its vertices. For example, let $B \subset V$. Then removal of the vertices in B from the graph G will yield the induced subgraph $G[V \setminus B]$. We may think of the induced subgraph $G[V \setminus B]$ as the part of G that remains when we remove B . Thus, for simplicity, we use the notation $G \setminus B$ to denote the induced subgraph $G[V \setminus B]$.

Definition 10. Two graphs G and H are said to be graph-homeomorphic if there is a graph isomorphism from a subdivision of G to a subdivision of H .

Recall that a graph G' is a subdivision of a graph G if we get G' by subdividing one or more edges in G : replace an edge $\{u, v\}$ with a pair of edges $\{u, w\}$ and $\{w, v\}$ where w is a new vertex. One can easily verify that graph-homeomorphism is an equivalence relation on the set of all graphs.

Definition 11. A graph G is said to be homeomorphically irreducible if whenever a graph H is homeomorphic to G , either $G = H$ or H is a subdivision of G .

Behzad and Chartrand show in [1] that in every equivalence class of homeomorphic graphs there exists a unique homeomorphically irreducible graph.

Definition 12. An embedding of a graph $G = (V, E)$ into a surface S is a function ϕ that maps distinct vertices of G to distinct points in S and edges of G to simple arcs (homomorphic images of $[0, 1]$) in S such that:

1. for every edge $e = \{v_1, v_2\}$ in E , the endpoints of the simple arc $\phi(e)$ are the points $\phi(v_1)$ and $\phi(v_2)$,
2. for all $e \in E$, and for all $v \in V$, $\phi(v)$ is not in the interior of $\phi(e)$,
3. for all $e_1, e_2 \in E$, $\phi(e_1)$ and $\phi(e_2)$ do not intersect at a point that is interior to either arc.

Note that if ϕ is an embedding of a graph G into a surface S , then the $\phi[G]$ is a simplicial 1-subcomplex of S .

Definition 13. The regions of an embedding ϕ of a graph G into a surface S are the connected components of $S \setminus \phi[G]$.

Minimal Separating Sets as Graph Embeddings

No 0-complex can separate a surface. Moreover, if M separates a surface S , then so does the 1-skeleton of M and it follows that every minimal separating set of a surface is a 1-complex. As such, a minimal separating set of a surface can be thought of as the image of a graph embedding into the surface.

Definition 14. If ϕ is an embedding of a graph G into a surface S and the image of ϕ is a minimal separating set of S then we say ϕ is a minimal separating embedding.

Note that if ϕ is an embedding of a graph G into S , and ψ is an embedding of a graph H into S , and if $\phi[G]$ and $\psi[H]$ are homeomorphic (in the topological sense) then the graphs G and H are graph-homeomorphic.

Definition 15. Suppose M is a minimal separating set of a surface S , and let $\mathcal{C}_M = \{G : \exists \text{ a graph embedding } \phi \text{ of } G \text{ into } S \text{ such that } \phi[G] = M\}$. Then \mathcal{C}_M contains a unique homeomorphically irreducible graph G_M . We say G_M is the graph underlying M .

Definition 16. We refer to a graph G as a minimal separating graph of S if G is the graph underlying some minimal separating set of S .

Definition 17. Let \mathbb{G}_g denote the set of all minimal separating graphs of an orientable surface of genus g .

Definition 18. We will denote the set of all minimal separating graphs $\cup_{g \geq 0} \mathbb{G}_g$ by \mathbb{G} .

We will classify the minimal separating sets of the orientable surfaces of genus $g \leq 3$ in various ways.

Definition 19. The least separated genus, $\gamma_-(G)$, of a graph G equals the minimal genus of the oriented surfaces in which G can be embedded in such a way that its embedding is a minimal separating set. It is ∞ if that set of (oriented) surfaces is empty.

We note here that theorem 7 shows that $\gamma_-(G) = \infty$ if and only if at least one vertex of G has odd degree or degree zero.

The Finitude of \mathbb{G}_g

By Euler's theorem we know that if ϕ is an embedding of G into S then $|V| - |E| + |R| \geq 2 - 2g$, where V is the vertex set of G , E is the edge set of G , R is the set of regions of the embedding, and g is the genus of S .

Lemma 1. If ϕ is a minimal separating embedding of G into S then $|R| = 2$.

Proof. Since $\phi[G]$ separates S , it is clear that $|R| \geq 2$. Now suppose $|R| > 2$. Since S is connected, there exist distinct regions f_1 and f_2 such that $\bar{f}_1 \cap \bar{f}_2 \neq \emptyset$. Since f_1 and f_2 are distinct regions, $f_1 \cap f_2 = \emptyset$. Thus $bd(\bar{f}_1) \cap bd(\bar{f}_2) \neq \emptyset$. Note that $bd(\bar{f}_1) \cap bd(\bar{f}_2) = \phi[\{e_1\} \cup \dots \cup \{e_n\}]$ for some $e_1, \dots, e_n \in E$. Let e be one of the e_1, \dots, e_n and let $x \in \text{int}(\phi(e))$. Then $x \in \phi[G]$, but $S \setminus (\phi[G] \setminus \{x\})$ has exactly one less connected component than $S \setminus \phi[G]$. Since $S \setminus \phi[G]$ has strictly more than 2 connected components, $S \setminus (\phi[G] \setminus \{x\})$ has at least 2 connected components. Thus $\phi[G] \setminus \{x\}$ separates S and $\phi[G]$ is not a minimal separating set. \square

Lemma 2. If ϕ is a minimal separating embedding of G into S and $e \in E$ then $\phi(e)$ is in the boundary of both regions of ϕ .

Proof. Suppose ϕ is a minimal separating embedding of a graph G into a surface S . It is clear that for every edge e , $\phi(e)$ must be in the boundary of at least one region. Suppose, by way of contradiction, that for some $e \in E$, $\phi(e)$ were in the boundary of exactly one of the two regions of ϕ . Then removal of an interior point x of $\phi(e)$ would not decrease the number of connected components. That is, $S \setminus (\phi[G] \setminus \{x\})$ has at least as many connected components as $S \setminus \phi[G]$. Hence $\phi([G] \setminus \{x\})$ is a separating set of S , and it follows that $\phi[G]$ is not *minimally* separating. \square

Theorem 1. A graph embedding is a minimal separating embedding if and only if the following conditions are satisfied:

1. The graph has no isolated vertices.
2. The embedding has exactly two regions.
3. The image of each edge is in the boundary of both regions of the embedding.

Proof. Suppose ϕ is a minimal separating embedding of G into S . Condition 1 is immediate. Lemmas 1 and 2 demonstrate conditions 2 and 3, respectively. So the forward direction is proved.

Now suppose ϕ is an embedding of G into S that satisfies the three conditions. Since condition 2 is satisfied, we know $\phi[G]$ separates S . To show that $\phi[G]$ is a *minimal* separating set, let $x \in \phi[G]$. Condition 1 implies that $x \in \phi[e]$ for some $e \in V$. By condition 3 we know that x is in the boundary of both regions of the embedding. It follows that $S \setminus (\phi[G] \setminus \{x\})$ is connected. \square

Lemma 3. *Suppose ϕ is a minimal separating embedding of G into S . Then $|E| - |V| \leq 2g$ where g is the genus of S .*

Proof. By Euler's theorem and lemma 1, $|V| - |E| + 2 \geq 2 - 2g$. Thus $|E| - |V| \leq 2g$ as desired. \square

Definition 20. *A graph homeomorphic to the graph consisting of one vertex with degree 2 and one edge (the cycle on one vertex) is called an isolated loop.*

Lemma 4. *For all g , and for all $G \in \mathbb{G}_g$, if v is a vertex of G and $\deg(v) = 2$, then v is an isolated loop.*

Proof. Suppose $g \geq 0$, and that $G \in \mathbb{G}_g$. Let v be a vertex of G such that $\deg(v) = 2$, and suppose that v is not an isolated loop. Then v is incident to precisely two edges, and so v can be smoothed. Thus G is not homeomorphically irreducible. Hence $G \notin \mathbb{G}_g$. \square

Lemma 5. *If $G \in \mathbb{G}$, then by Lemma all vertices of G have even degree.*

Proof. Let v be a vertex and consider a neighborhood around $\phi(v)$. The edge-ends emanating from v divide the neighborhood into $\deg(v)$ subdivisions. By Lemma 1, there are exactly 2 regions of ϕ and each of these subdivisions must be a subset of exactly one of the two regions. By Lemma 2, each of the edge ends must be in the boundary of both regions. If $\deg(v)$ were odd then there would exist an edge end emanating from v that was in the boundary of only one of the two regions. Hence $\deg(v)$ is not odd. \square

In what follows we denote the set of isolated loops of a graph G by B . If G has no isolated loops then $B = \emptyset$. The graph G minus the isolated loops is denoted by $G \setminus B$, and its set of vertices and set of edges are denoted by $V(G \setminus B)$ and $E(G \setminus B)$, respectively. The number of components of B will be denoted by $|B|$.

Lemma 6. *For all $g \geq 0$, and for all $G \in \mathbb{G}_g$, $2|V(G \setminus B)| \leq |E(G \setminus B)|$.*

Proof. Let $g \geq 0$, and let $G \in \mathbb{G}_g$. Let B be the set of isolated loops of G . By Lemmas 4 and 5, and the observation that minimal separating graphs do not contain vertices of degree zero, we have that $\deg(v) \geq 4$ for all $v \in V(G \setminus B)$. Thus we have $4|V(G \setminus B)| + 2|B| \leq \left[\sum_{v \in V \setminus B} \deg(v) \right] + 2|B| = 2|E(G \setminus B)| + 2|B|$. Thus $2|V(G \setminus B)| \leq |E(G \setminus B)|$. \square

Lemma 7. *Let B be the set of isolated loops of a graph $G \in \mathbb{G}_g$. Either the graph $G \setminus B$ is empty or it is a minimal separating graph of a surface of genus $g - |B|$, where $|B|$ is the number of components of B .*

Proof. Suppose G is a minimal separating graph for a surface S of genus g , and let ϕ be a minimal separating embedding of G into S . It will suffice to show that if L is any isolated loop of G , and $G \setminus L$ is not empty, then there exists an embedding of $G \setminus L$ into a surface of genus $g - 1$.

Let L be any isolated loop of G . Either $\phi[L]$ is simple (ie contractible to a point), or it is not. If $\phi[L]$ is contractible to a point, then L separates S , and it follows that $G \setminus L$ is empty.

Suppose $G \setminus L$ is not empty. Then $\phi[L]$ is not simple. So $\phi[L]$ must circumnavigate a handle of S . We will construct a minimal separating embedding of the graph $G \setminus L$ into a surface of genus $g - 1$ using surgery.

Note that ϕ satisfies the three conditions of theorem 1. A closed neighborhood of $\phi[L]$ is a cylinder C . Remove C from S . Since the Euler characteristic of a cylinder is zero, we have $\chi_S = \chi_{S \setminus C}$. Note that by condition 3 of theorem 1, this has the result of creating exactly two boundary components (one in each region of $S \setminus \phi[G]$). Glue one disk into each of the two components. Gluing in a disk adds 1 to the characteristic of the surface. What remains is an embedding of the graph $G \setminus L$ into a surface of surface of genus $g - 1$. Note that this embedding is a minimal separating embedding because it satisfies the three requirements of theorem 1 – for it inherited these properties from ϕ . \square

Colloquially, the above lemma states that, in minimal separating graphs, the only role played by isolated loops is the role of handle-cutter. The sole exception is the contractible loop. And in that case the isolated loop is the minimal separating graph itself.

Lemma 8. *For all g , and for all $G \in \mathbb{G}_g$, if B is the set of isolated loops of G then $|B| \leq g + 1$.*

Proof. Suppose $G \in \mathbb{G}_g$, and \tilde{B} is a set of g isolated loops of G . Then by Lemma 7, the graph $G \setminus \tilde{B}$ has a minimal separating embedding in a surface of genus 0 (a sphere). Thus $G \setminus \tilde{B}$ must be an isolated loop. \square

Lemma 9. *Let $G \in \mathbb{G}_g$, and suppose that G contains a collection of $g + 1$ non-intersecting cycles. Then G is the graph consisting of $g + 1$ isolated loops.*

Proof. As in the proof of lemma 7, remove a closed neighborhood C of g of the non-intersecting cycles from the surface S and glue to the resulting boundary components of $S \setminus C$. The resulting surface is a sphere. Since G contained another non-intersecting cycle, the sphere also does, separating the sphere. It follows that the original $g + 1$ cycles in G separate S . By minimality G must be equal to that collection of cycles. \square

Lemma 10. *For all $g \geq 1$, and for all $G \in \mathbb{G}_g$, if B is the set of isolated loops of G and $G \setminus B$ is not empty, then $0 \leq |V(G \setminus B)| \leq 2g - 2|B|$.*

Proof. Suppose $g \geq 1$, and let $G \in \mathbb{G}_g$. Let B be the set of isolated loops of G .

If $G \setminus B$ is not empty then, by lemma 7, the graph $G \setminus B$ is in $\mathbb{G}_{\hat{g}}$, where $\hat{g} = g - |B|$. It follows by lemma 3 that $|E(G \setminus B)| - |V(G \setminus B)| \leq 2\hat{g}$. By Lemma 6 we have $2|V(G \setminus B)| \leq |E(G \setminus B)|$. Thus we find $|V(G \setminus B)| \leq |E(G \setminus B)| - |V(G \setminus B)| \leq 2\hat{g} = 2(g - |B|)$ as desired. \square

Theorem 2. *For all g , \mathbb{G}_g is finite.*

Proof. Fix g , and let $G \in \mathbb{G}_g$. Let B be the set of isolated loops of G . By lemmas 10 and 8, $0 \leq |V(G)| = |V(G \setminus B)| + |B| \leq 2g - |B|$. This puts an upper bound on the size of $|V(G)|$ for any graph G in \mathbb{G}_g . This, in combination with the statement of Lemma 3, provides an upper bound on $|E|$. In particular, $0 < |E(G)| \leq 2g + |V(G)|$. Thus, \mathbb{G}_g is finite. \square

Definition 21. For $g \geq 1$ define \mathbb{C}_g (the ‘candidate graphs’) to be the set of all graphs G satisfying the following properties:

1. If $G \setminus B$ is not empty, then $0 \leq |V(G)| \leq 2g - |B|$,
2. $0 < |E(G)| \leq 2g + |V(G)|$
3. All vertices of G have even degree greater than zero
4. The only vertices of G with degree two are isolated
5. G has at most g non-intersecting cycles, unless it equals $g + 1$ isolated loops.

We refer to \mathbb{C}_g as the set of candidate graphs for genus g .

It is clear that for all $g \geq 1$, \mathbb{C}_g is finite and $\mathbb{G}_g \subset \mathbb{C}_g$. For low genus surfaces, \mathbb{C}_g can be generated quite easily using a contemporary computer. To decide which members of \mathbb{C}_g are also in \mathbb{G}_g , we provide a method to test whether a given graph in \mathbb{C}_g has a minimal separating embedding in a surface of genus g .

Rotation Systems

Definition 22. A graph embedding induces, for each vertex v , a cyclic ordering of edges incident to v . This ordering is referred to as the rotation at v , and is unique up to a cyclic permutation of the edges. Without loss of generality, we will always consider cyclic orderings in the clockwise direction.

Definition 23. A rotation system for a graph G consists of an unordered set \mathcal{R} of the vertices $\{v_i\}$ of G together with a cyclically ordered list of all the edges e_j that are incident to v_i . Note that each edge must occur exactly twice in a rotation system.

Remark: This corresponds to the notion of *pure rotation system* as given in [3]. Note that one graph may have many embeddings, leading to non-equivalent rotation systems. For instance the graphs in Figure 1 have rotation systems given in the table at the end of this article as graph # 1, 2, 3, and 8.

Definition 24. A graph embedding whose regions are homeomorphic to open disks is called a *cellular embedding*.

Cellular embeddings with the same rotation system are considered equivalent. As there are only finitely many possible rotations at each vertex, and only finitely many vertices in a graph, we know there are a finite number of possible rotation systems for a given graph. Thus the infinite set of all possible cellular embeddings of a graph into a surface can be partitioned into equivalence classes of finitely many rotation systems. The following is taken from [3].

Theorem 3. *Every rotation system on a graph G defines a unique (up to orientation preserving homeomorphisms) locally oriented cellular graph embedding $G \rightarrow S$. Conversely every locally oriented cellular embedding defines a rotation system.*

For the calculations that follow, it will be useful to consider a closed neighborhood around the image of a graph embedding.

Definition 25. *Let ϕ be an embedding of a graph G into a surface S . We surround the image of each vertex with a closed disk, referred to as a 0-band, and surround the image of each edge with a thin band, referred to as a 1-band. The union of 0-bands and 1-bands forms a closed neighborhood in S that preserves the shape of the graph G . We refer to the union of the 0-bands and 1-bands (omitting the rest of the surface) as the reduced band decomposition of ϕ .*

For a more thorough discussion of rotation systems and reduced band decompositions see [3].

Definition 26. *Every region of an embedding ϕ of a graph G into a surface S is bounded by the image of a closed walk (ie a topological circle). This closed walk is unique (up to a cyclic permutation) and the ordered list of edges in the walk is referred to as the boundary walk of the region.*

Definition 27. *Let \mathcal{B} be the reduced band decomposition corresponding to a graph embedding ϕ of a graph G into a surface S , and let b be one of the boundary components of \mathcal{B} . Then $\langle b \rangle$ denotes the set of edges of G that appear in the boundary walk of b .*

Definition 28. *Let \mathcal{R} be a rotation system for an embedding of a graph G with no isolated vertices. Let b_1, b_2, \dots, b_n be the boundary components of the reduced band decomposition corresponding to \mathcal{R} . We refer to \mathcal{R} as a two-sided rotation system if there exists a partition of the set $\{b_1, b_2, \dots, b_n\}$ into two sets of closed boundary walks, B_1 and B_2 , such that for $\forall e \in E$, $e \in \bigcup_{b \in B_1} \langle b \rangle$ and $e \in \bigcup_{b \in B_2} \langle b \rangle$.*

The cardinalities $|B_1|$ and $|B_2|$ will be denoted by n_1 and n_2 .

Colloquially speaking, a two-sided rotation system \mathcal{R} is a one for which:

1. All vertices of the underlying graph G have even degree greater than zero and
2. There exists an embedding ϕ of G with the following properties:
 - (a) The rotation system corresponding to ϕ is \mathcal{R}
 - (b) The embedding ϕ has exactly two regions, one with boundary B_1 and the other with boundary B_2 .
 - (c) The image of each edge can be found in the boundary of each region.

Identifying Minimal Separating Graphs

Upon fixing a genus g we can create the finite set \mathbb{C}_g of candidate graphs. We know that the set \mathbb{G}_g of minimal separating graphs is contained in \mathbb{C}_g . Moreover,

for each graph $G \in \mathbb{C}_g$ we can generate the finite set \mathbf{R}_G of all rotation systems on G .

Now suppose $R \in \mathbf{R}_G$. We want a way to test whether there exists a minimal separating embedding ϕ from G into S such that:

1. The rotation system corresponding to ϕ is R and
2. The surface S has genus g .

Let's begin with the following useful observation.

Corollary 1. $\mathbb{G}_0 \subset \mathbb{G}_1 \subset \mathbb{G}_2 \subset \dots$

Proof. Let $G \in \mathbb{G}_k$ for some $k \geq 0$. Let S_k be a surface of genus k and let ϕ_k be a minimal separating embedding of G into S_k . If we attach a handle to one of the two regions of ϕ_k , we increase the genus of the surface by one while the three conditions of theorem 1 remain satisfied. The result is a minimal separating embedding ϕ_{k+1} of G into a surface S_{k+1} of genus $k+1$. \square

In light of theorem 1, it follows that minimal separating graph embeddings are exactly those with two-sided rotation systems. This is made rigorous with the following theorem.

Theorem 4. *There exists a minimal separating embedding ϕ of a graph G with corresponding rotation system \mathcal{R} if and only if \mathcal{R} is a two-sided rotation system.*

Proof. Suppose there exists a minimal separating embedding ϕ with corresponding rotation system \mathcal{R} . Then by condition 2 of theorem 1, there are exactly two regions of ϕ . Let r_1 and r_2 be the regions of ϕ . Note that every boundary component of \mathcal{B} lies in exactly one of the two regions of ϕ . So form the desired partition of $\{b_1, b_2, \dots, b_n\}$ by defining B_1 to be the set of boundary components of \mathcal{B} that lie in region r_1 and defining B_2 analogously. Then condition 3 of theorem 1 tell us that $\forall e \in E, \forall i \in \{1, 2\}, \left(e \in \bigcup_{b \in B_i} \langle b \rangle \right)$, and the forward direction is proved.

Now suppose we have a two-sided rotation system so that the set $\{b_1, b_2, \dots, b_n\}$ can be partitioned into two boundaries, B_1 and B_2 , such that for $\forall e \in E, \forall i \in \{1, 2\}, \left(e \in \bigcup_{b \in B_i} \langle b \rangle \right)$. The number of connected components in B_1 (B_2) will be denoted by n_1 (n_2). We now construct a minimal separating (cellular) embedding ϕ with corresponding rotation system \mathcal{R} using surgery.

By theorem 3, \mathcal{R} defines a cellular embedding $\phi : G \rightarrow S$ in a surface S . This embedding defines (by definition 24) a reduced band decomposition \mathcal{D} with boundary components (definition 28) that are (disjoint) topological circles. These circles come in two sets, those that can be retracted to B_1 and which we will call \tilde{B}_1 and those that can be retracted to B_2 (called \tilde{B}_2).

Now consider two spheres, A_1 and A_2 . Remove n_1 open discs from A_1 and name the result \overline{A}_1 . Construct \overline{A}_2 similarly. We identify the boundary components of \tilde{B}_i to the boundary components of \overline{A}_i through a bijection ψ_i :

$$\psi_i : \partial(\overline{A}_i) \rightarrow \tilde{B}_i \quad (1)$$

The result is a surface S as a connected sum of the reduced band decomposition \mathcal{D} with the n_i -punctured spheres \overline{A}_i :

$$S = \mathcal{D} \# \overline{A}_1 \# \overline{A}_2 \quad (2)$$

By construction, there are exactly two regions of the embedding (the regions corresponding to \overline{A}_1 and \overline{A}_2), and every edge is on the boundary of both regions. So by theorem 1, the embedding is a minimal separating embedding. \square

Given an embedding ϕ of a graph G , the boundary components of the reduced band decomposition can be computed from the rotation system by a simple combinatorial algorithm called face tracing (section 3.2.6 of [3]).

Lemma 11. *Given the representation*

$$R = \begin{array}{rcl} v_0 & : & a \, x_1 \cdots x_p \, a \, x_{p+1} \cdots x_k \\ v_1 & : & \cdots \\ \vdots & & \end{array}$$

of the rotation system \mathcal{R} of a minimally separating embedding of a graph G it follows that p and k are even.

Proof. By Lemma 5, p is even iff k is even. We show that p is even.

In the embedding edge a must separate a boundary component $b_1 \in B_1$ from a boundary component $b_2 \in B_2$. Orient edge a and travel with the orientation along a . The boundary component at your right side (with respect to the orientation) must always point towards the same component. It follows that the corner (x_k, a) corresponds to the same boundary component as the corner (a, x_{p+1}) . The same is true for the other two corners involving edge ends of a . Since the corners belonging to boundary component in B_1 must alternate with those belonging to B_2 , we conclude that p is even. \square

Theorem 5. *A graph G is in \mathbb{G}_g for some g if and only if there exists a two-sided rotation system \mathcal{R} on G such that $g \geq \frac{|E| - |V| + n_1 + n_2}{2} - 1$ (where n_1 and n_2 are the numbers of boundary components corresponding to regions 1 and 2 of the reduced band decomposition corresponding to \mathcal{R}).*

Proof. G is in \mathbb{G}_g if and only if there exists a two-sided rotation system \mathcal{R} by theorem 4. Equivalently the embedding in the surface S with genus g admits a reduced band decomposition \mathcal{D} with n_1 bordering on the component A_1 of $S \setminus \mathcal{D}$ and n_2 boundary components bordering of the component A_2 . Each of these components A_i has itself genus g_i .

The genus of S is the genus of the connected sum in Equation 2 in the proof of theorem 4. Thus $\chi_S = \chi(\overline{A}_1) + \chi(\overline{A}_2) + \chi(D)$. Since the characteristic of an n -punctured surface of genus g_i equals $2 - 2g_i - n_i$, we have

$$\chi_S = 2 - 2g_1 - n_1 + 2 - 2g_2 - n_2 + |V| - |E| \quad (3)$$

Solving for the genus g_S of S gives:

$$g_S = \frac{|E| - |V| + n_1 + n_2}{2} - 1 + g_1 + g_2 \quad (4)$$

Clearly g is minimized when $g_1 = g_2 = 0$. \square

Definition 29. The minimally separating embedding of G is irreducible if equality holds in Theorem 5 holds (ie if $g_1 = g_2 = 0$ in Equation 4).

Lemma 12. If $G \in \mathbb{G}_g$ consists of two disjoint nonempty subgraphs \underline{G} and \overline{G} we have each of the subgraphs as elements of $\bigcup_{g' < g} \mathbb{G}_{g'}$.

Proof. It is sufficient to prove the lemma for \underline{G} . Suppose G embeds as a minimal separating set in the surface S . Denote the quantities corresponding to the disjoint subgraphs by underbars and overbars, ie $|\underline{E}|$ is the number of edges in \underline{G} and so forth. Let \underline{C} be a closed neighborhood of $\phi[\underline{G}]$. Then ϕ restricted to \underline{C} satisfies theorem 1. Remove \underline{C} from S . Let \underline{A}_1 and \underline{A}_2 be the \underline{n}_1 and \underline{n}_2 punctured spheres. We denote by \underline{S} the connected sum

$$\underline{S} = \underline{C} \# \underline{A}_1 \# \underline{A}_2$$

The genus of this surface according to equation 4 equals

$$g_{\underline{S}} = \frac{|\underline{E}| - |\underline{V}| + \underline{n}_1 + \underline{n}_2}{2} - 1$$

By the same reasoning the genus of the original surface S equals

$$\begin{aligned} g_S &= \frac{|E| - |V| + n_1 + n_2}{2} - 1 + g_1 + g_2 \\ &= \frac{|\overline{E}| - |\overline{V}| + \overline{n}_1 + \overline{n}_2 + |\underline{E}| - |\underline{V}| + \underline{n}_1 + \underline{n}_2}{2} - 1 + g_1 + g_2 \end{aligned}$$

Lemma 4 implies that $|\overline{E}| - |\overline{V}| \geq 0$ and lemma 2 that \overline{n}_1 and \overline{n}_2 are at least 1. Thus $g_S \geq g_{\underline{S}} + 1$. \square

Lemma 13. If \underline{G} and \overline{G} have least separated genus \underline{g} and \overline{g} , respectively, then their disjoint union has least separated genus $\underline{g} + \overline{g} + 1$

Proof. Let $\underline{\mathcal{R}}$ be the rotation system for \underline{G} that minimizes the number of boundary walks $\underline{n}_1 + \underline{n}_2$. The least separated genus \underline{g} satisfies:

$$\underline{g} = \frac{|\underline{E}| - |\underline{V}| + \underline{n}_1 + \underline{n}_2}{2} - 1$$

The same holds for \overline{G} . The rotation system for the disjoint union of \underline{G} and \overline{G} , is the union of their two rotation systems $\mathcal{R} = \underline{\mathcal{R}} \cup \overline{\mathcal{R}}$. The boundary walks of \mathcal{R} are exactly the union those of in $\underline{\mathcal{R}}$ and those in $\overline{\mathcal{R}}$. Therefore the minimum number of boundary walks in \mathcal{R} equals $\underline{n}_1 + \underline{n}_2 + \overline{n}_1 + \overline{n}_2$. And thus the least separated genus g of G satisfies:

$$g = \frac{|\overline{E}| - |\overline{V}| + \overline{n}_1 + \overline{n}_2 + |\underline{E}| - |\underline{V}| + \underline{n}_1 + \underline{n}_2}{2} - 1$$

which implies the result. \square

The Results

Given a genus g , we compute three sets associated with it.

$$\begin{aligned}\mathbb{I}_g &= \{\text{Connected graphs whose least separated genus equals } g\} \\ \mathbb{L}_g &= \{\text{Graphs whose least separated genus equals } g\} \\ \mathbb{G}_g &= \{\text{Graphs that have a minimal separating embedding in genus } g\}\end{aligned}$$

Proposition 1. *We have: $|\mathbb{I}_2| = 17$ and $|\mathbb{I}_3| = 161$.*

Proof. This count was computer generated. The counting program follows the following steps.

1. Generate the finite set of connected graphs in \mathbb{C}_g of candidate graphs for genus g (see definition 21, it is important to note that condition 5 is computationally expensive so a weaker version is used which looks for non-intersecting loops instead of non-intersecting cycles.).
2. For each connected graph G in \mathbb{C}_g generate the finite set of rotation systems of G satisfying Lemma 11 up to cyclic permutation.
3. For each rotation system found in step 2, check whether it is two-sided and satisfies $g = \frac{|E| - |V| + n_1 + n_2}{2} - 1$ (see theorem 5).

The graph G of step 2 is in \mathbb{I}_g if and only if any rotation system tests affirmatively in step 3 (and did not appear in a surface with lower genus). We keep a running list of the graphs G that pass the test, as well as the successful rotation systems. In genus 2, we determine all rotation systems that pass the test (see the table at the end of the article). In genus 3, we stop if we have 1 successful rotation system for a graph. \square

Using the computer-generated count and some combinatorics yields the main result.

Theorem 6. *The cardinalities of \mathbb{I}_g , \mathbb{L}_g , and \mathbb{G}_g for $g \leq 3$ are as follows.*

Genus	\mathbb{I}_g	\mathbb{L}_g	\mathbb{G}_g
0	1	1	1
1	3	4	5
2	17	21	26
3	161	191	217

Proof. The numbers for genera 0 and 1 are discussed in [2] and [5]. The cardinalities of $|\mathbb{I}_2|$ and $|\mathbb{I}_3|$ are given in Proposition 1. We show that the counts for \mathbb{L}_g for \mathbb{G}_g can be expressed in terms of those for \mathbb{I}_g .

To obtain $|\mathbb{L}_g|$ from $|\mathbb{I}_i|_{i \leq g}$, we must add to $|\mathbb{I}_g|$ graphs G with more than one component and such that $\gamma_-(G) = g$. According to Lemma 12, each component of such a graph G separates a lower genus. Lemma 13 then implies that:

$$\begin{aligned}|\mathbb{L}_2| &= |\mathbb{I}_0||\mathbb{I}_0||\mathbb{I}_0| + |\mathbb{I}_0||\mathbb{I}_1| + |\mathbb{I}_2| \\ |\mathbb{L}_3| &= |\mathbb{I}_0||\mathbb{I}_0||\mathbb{I}_0||\mathbb{I}_0| + |\mathbb{I}_0||\mathbb{I}_0||\mathbb{I}_1| + |\mathbb{I}_0||\mathbb{I}_2| + |\mathbb{I}_1||\mathbb{I}_1| + |\mathbb{I}_3|\end{aligned}$$

Any graph that minimally separates genus $g - 1$, also minimally separates genus g (just add a handle to one of the components of the complement of the graph in a minimal separating embedding). Thus the total count of all minimal separating graphs in genus g can be computed as follows

$$|\mathbb{G}_g| = \sum_{i=0}^g |\mathbb{L}_i|$$

□

The graphs in \mathbb{L}_0 , \mathbb{L}_1 , and \mathbb{L}_2 are illustrated in Figures 1 through 5. The rotation systems that allow minimal separating embeddings in genus 2 are given in the table at the end of this article.

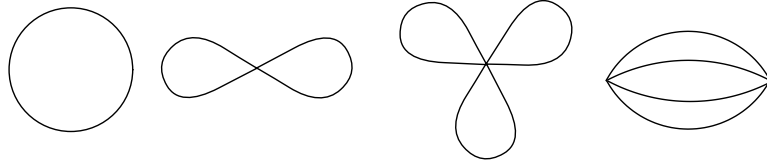


Figure 1: \mathbb{L}_0 consists of one graph (the circle). The other three constitute the set \mathbb{L}_1 . From left to right, respectively graphs 1, 2, 3, and 8 of the table.

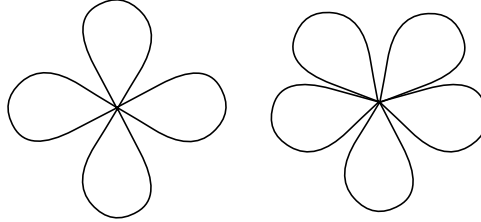


Figure 2: Graphs in \mathbb{L}_2 with one vertex. From left to right, respectively graphs 4, and 5 of the table.

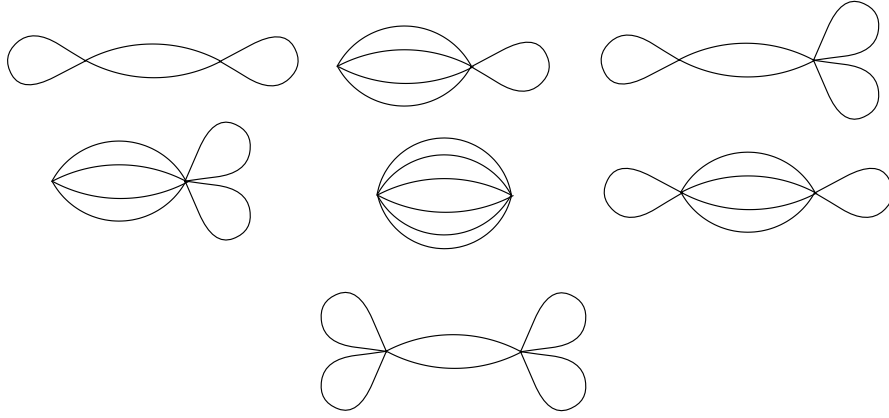


Figure 3: *Graphs in \mathbb{I}_2 with two vertices. From left to right first row, respectively graphs 10, 11, and 12 of the table. On the second row, graphs 13, 14 , and 15. Last row, graph 16.*

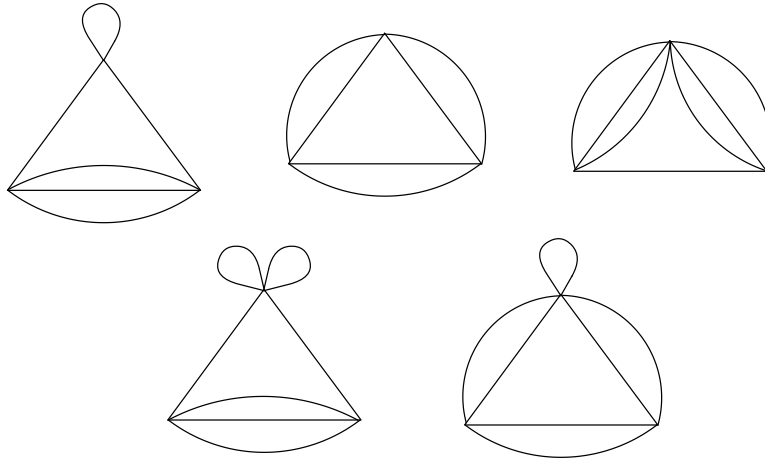


Figure 4: *Graphs in \mathbb{I}_2 with three vertices. From left to right, top row, graphs 19, 20, and 21; below graphs 22 and 23.*

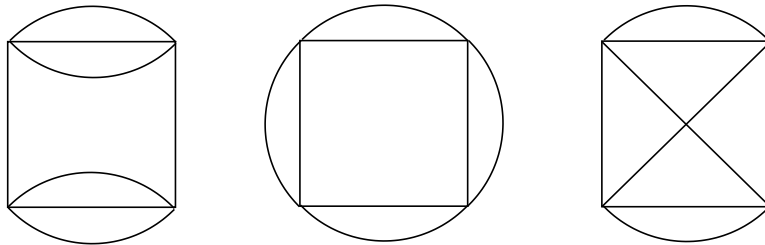


Figure 5: *Graphs in \mathbb{I}_2 with four vertices. From left to right, respectively graphs 24, 25, and 26.*

Final Remarks

The algorithm in Proposition 1 is computationally extremely expensive. The number of (multi)graphs that satisfy Definition 21 in genus 3 is very large. In addition, for each graph G there are in principle $\prod_{v \in V(G)} (\deg(v)!)^2$ possible rotation systems. Even using the techniques mentioned in step 2 of that Proposition to reduce this number, it still grows far too quickly for our current algorithm to be used for genus $g > 3$ on a contemporary desktop computer.

The graphs for $g \leq 2$ are listed and numbered in the Appendix along with their two-sided rotation systems. The set \mathbb{G}_1 consists of graphs 1, 2, 3, 6, and 8 (in agreement with [2]). The set \mathbb{G}_2 consists of graphs 1 through 26.

We end with some interesting observations that were however not used in the computation of the cardinalities discussed in the last section. The first is that a graph minimally separates in some genus if and only if all its vertices have even degree.

Theorem 7. *A graph G is in \mathbb{G} if and only if all its vertices have positive, even degree.*

Proof. One direction is proved by part 1 of theorem 1 and lemma 5. So we need to prove that if G has all degrees even and positive, then it minimally separates. By lemma 13, it is sufficient to prove this for connected graphs.

Let G be a connected non-empty graph of even degree. Then G admits an Eulerian circuit γ consisting of all its n edges. Label these edges e_1, e_2, \dots, e_n , in the order in which they occur in the circuit γ . Denote the first vertex of e_i , as we traverse γ , by v_i . Note that v_i maps $\{1, \dots, n\}$ onto the vertex set, but it is not generally a bijection.

Construct a rotation system \mathcal{R} for G as follows. At v_1 add the edge pair $e_n - e_1$. For all $k \in \{2, \dots, n\}$, at v_k add the edge pair $e_{k-1} - e_k$. No matter what the ordering of the pairs is at any one vertex in the resulting rotation system, one boundary walk is given by the edge sequence γ , which contains every edge exactly once. Thus the set of all boundary walks can be partitioned into $B_1 = \gamma$ on the one hand, and B_2 is the remainder of the boundary walks. Since all edges must occur twice, B_2 also contains every edge exactly once. And so \mathcal{R} is two-sided. \square

For our final remark we need the following Definition (compare with Definition 19). Recall the definition of irreducible embedding (Definition 29).

Definition 30. *The largest irreducibly separated genus, $\gamma_+(G)$, of a graph G equals the maximal genus of the oriented surfaces in which G can be irreducibly embedded as a minimally separating set. It is ∞ if that set of (oriented) surfaces is either empty or if their genus has no upper bound.*

Denote by \mathbb{G}_E the graphs all of whose vertices have even degree. Thus if $G \in \mathbb{G}_E$, then $\gamma_-(G)$ and $\gamma_+(G)$ are finite by the previous Theorem. So for these graphs there are always minimally separating embeddings. There is an interesting relation between these quantities on the one hand and the *minimum genus* $\gamma_{\min}(G)$ and the *maximum genus* $\gamma_{\max}(G)$ on the other. While the problem of deciding the minimum genus of a graph G is NP-complete ([4]), there are explicit estimates for the maximum genus ([6]). Thus the following Theorem enables us to give an explicit lower bound for $\gamma_-(G)$.

Theorem 8. For graphs G in \mathbb{G}_E we have the following relations:

$$\begin{aligned}\gamma_-(G) &\geq |E| - |V| - \gamma_{\max}(G) \\ \gamma_+(G) &\leq |E| - |V| - \gamma_{\min}(G)\end{aligned}$$

Proof. Fix a graph $G \in \mathbb{G}_E$. Let \mathcal{R}_G be the set set of rotation system for that graph, and $\mathcal{R}_G^* \subseteq \mathcal{R}_G$ the set of two-sided rotation systems for that graph. Every two-sided rotation system R defines an irreducible minimally separating embedding in a surface S whose genus g_S satisfies (Equations 3 and 4 with $g_1 = g_2 = 0$):

$$g_S = \frac{|E| - |V| + N}{2} - 1$$

On the other hand every rotation system R for G defines a cellular embedding in a surface C by gluing a disk onto each of the N boundary components of R . The genus g_C of C satisfies:

$$|V| - |E| + N = 2 - 2g_C \quad \Rightarrow \quad |E| - |V| - g_C = \frac{|E| - |V| + N}{2} - 1$$

The last two equations imply:

$$\gamma_-(G) = \min_{R \in \mathcal{R}_G^*} \frac{|E| - |V| + N}{2} - 1 \geq \min_{R \in \mathcal{R}_G} \frac{|E| - |V| + N}{2} - 1 = |E| - |V| - \gamma_{\max}(G)$$

The second relation is similarly proved by taking the maximum. \square

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# of Graph	Rotation System	(n_1, n_2)	(g_1, g_2)
1	$v_0 : 00$	(1,1)	(0,2), (1,1)
2	$v_0 : 0011$	(1,2)	(0,1), (1,0)
3	$v_0 : 001122$	(1,3)	(0,0)
	$v_0 : 011022$	(2,2)	(0,0)
	$v_0 : 012012$	(1,1)	(0,1)
4	$v_0 : 00123123$	(1,2)	(0,0)
	$v_0 : 01203213$	(1,2)	(0,0)
5	$v_0 : 0120134234$	(1,1)	(0,0)
	$v_0 : 0120324314$	(1,1)	(0,0)
	$v_0 : 0123401234$	(1,1)	(0,0)
6	$v_0 : 00$ $v_1 : 11$	(2,2)	(0,1)
7	$v_0 : 0011$ $v_1 : 22$	(2,3)	(0,0)
8	$v_0 : 0123$ $v_1 : 0123$	(1,1)	(0,1)
	$v_0 : 0123$ $v_1 : 0321$	(2,2)	(0,0)
9	$v_0 : 00$ $v_1 : 123123$	(2,2)	(0,0)
10	$v_0 : 0012$ $v_1 : 1332$	(2,2)	(0,0)
	$v_0 : 0012$ $v_1 : 1233$	(2,2)	(0,0)
11	$v_0 : 0123$ $v_1 : 042143$	(1,2)	(0,0)
	$v_0 : 0123$ $v_1 : 044123$	(1,2)	(0,0)
12	$v_0 : 0012$ $v_1 : 134234$	(1,2)	(0,0)
13	$v_0 : 0123$ $v_1 : 04512345$	(1,1)	(0,0)
	$v_0 : 0123$ $v_1 : 04534125$	(1,1)	(0,0)
14	$v_0 : 012345$ $v_1 : 012345$	(1,1)	(0,0)
	$v_0 : 012345$ $v_1 : 014523$	(1,1)	(0,0)
15	$v_0 : 012034$ $v_1 : 154253$	(1,1)	(0,0)
16	$v_0 : 012013$ $v_1 : 245345$	(1,1)	(0,0)

# of Graph	Rotation System	(n_1, n_2)	(g_1, g_2)
17	$v_0 : 00$ $v_1 : 11$ $v_2 : 22$	(3,3)	(0,0)
18	$v_0 : 00$ $v_1 : 1234$ $v_2 : 1234$	(2,2)	(0,0)
19	$v_0 : 0012$ $v_1 : 1345$ $v_2 : 2345$	(1,2)	(0,0)
20	$v_0 : 0123$ $v_1 : 0451$ $v_2 : 2345$	(1,2)	(0,0)
	$v_0 : 0213$ $v_1 : 0415$ $v_2 : 2435$	(1,2)	(0,0)
21	$v_0 : 0123$ $v_1 : 0456$ $v_2 : 123456$	(1,1)	(0,0)
	$v_0 : 0123$ $v_1 : 0456$ $v_2 : 125634$	(1,1)	(0,0)
22	$v_0 : 0123$ $v_1 : 1236$ $v_2 : 045645$	(1,1)	(0,0)
23	$v_0 : 0123$ $v_1 : 0145$ $v_2 : 265364$	(1,1)	(0,0)
	$v_0 : 0213$ $v_1 : 0415$ $v_2 : 263564$	(1,1)	(0,0)
24	$v_0 : 0123$ $v_1 : 4567$ $v_2 : 0456$ $v_3 : 1237$	(1,1)	(0,0)
25	$v_0 : 0213$ $v_1 : 4657$ $v_2 : 0415$ $v_3 : 2736$	(1,1)	(0,0)
26	$v_0 : 0123$ $v_1 : 0456$ $v_2 : 1457$ $v_3 : 2367$	(1,1)	(0,0)
	$v_0 : 0123$ $v_1 : 0465$ $v_2 : 1475$ $v_3 : 2367$	(1,1)	(0,0)